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Rigorous Bounds of the Lyapunov Exponents of the One-Dimensional Random Ising Model

Michele Pasquini,[1] Giovanni Paladin,[1] and Maurizio Serva[2]

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We find analytic upper and lower bounds of the Lyapunov exponents of the product of random matrices related to the one-dimensional disordered Ising model, using a deterministic map which transforms the original system into a new one with smaller average couplings and magnetic fields. The iteration of the map gives bounds which estimate the Lyapunov exponents with increasing accuracy. We prove, in fact, that both the upper and the lower bounds converge to the Lyapunov exponents in the limit of infinite iterations of the map. A formal expression of the Lyapunov exponents is thus obtained in terms of the limit of a sequence. Our results allow us to introduce a new numerical procedure for the computation of the Lyapunov exponents which has a precision higher than Monte Carlo simulations.

KEY WORDS: Random Ising model; one-dimensional; Lyapunov exponents; rigorous bounds.

INTRODUCTION

In recent years products of random matrices have become a contact point between different fields, such as disordered and chaotic dynamical systems. In that context the Lyapunov characteristic exponents represent physically important quantities (for a general review see refs. 1 and 2). However, even in simple situations, the direct calculation of the Lyapunov spectrum is a very difficult problem, which is exactly solved only for few particular cases. On the other hand, many different approaches have been developed in order to give analytic approximations of the Lyapunov exponents.[2]

The one-dimensional random Ising model is a disordered system where the explicit form of the Lyapunov exponents (related to the free energy and

[1] Dipartimento di Fisica, Università dell'Aquila, I-67100 Coppito, L'Aquila, Italy.
[2] Dipartimento di Matematica, Università dell'Aquila, I-67100 Coppito, L'Aquila, Italy.

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to the rate of the correlation decay) is not known, in spite of the simple expression of the matrices involved. Recently, we have proposed[3] an analytic solution of the model by the determination of the fixed point of a deterministic map which reduces the system to a new one-dimensional Ising model, with smaller average values of the random quenched variables. In this paper, we modify that map to make it appropriate to prove various statements which have been only conjectured on a heuristic basis in ref. 3. In particular, our new map transforms the system into a new one which always has real couplings and fields, in contrast with ref. 3.

The main results of the paper are thus the following:

1. Rigorous upper and lower bounds of the Lyapunov exponents whose precision increases with the number of map iterations. From a numerical point of view, we get very accurate estimates (better than those obtained by other numerical methods) in the case of discrete disorder distributions.
2. The proof of the convergence of these bounds toward the two Lyapunov exponents in the limit of infinite map iterations.
3. The formal expression of the Lyapunov exponents can be also characterized via the convergence of the map toward a fixed point before performing the thermodynamic limit, in a restricted ensemble of disorder realizations. Then, the infinite-volume limit can be performed since that ensemble is constructed in such a way that it has full probability measure.

The paper is organized as follows:

In Section 1, there is a brief introduction to the random Ising model. In Section 2, we describe the deterministic map which reduces the average values of the couplings and of the magnetic fields of the Ising model. In Section 3, we find a formal expression of the Lyapunov exponents as functions of a coupling J^*, related to the fixed point of the map. In Section 4, rigorous upper and lower bounds of the Lyapunov exponents are obtained by a finite number n of iterations of the map. These bounds can be exactly computed and in limit $n \to \infty$ they converge to the Lyapunov exponents. They are explicitly calculated for a dichotomic distribution of the magnetic fields. In Appendix A, we show that the formal expression of the Lyapunov exponents can be also obtained in a rigorous context by inverting the thermodynamic limit and the limit over the map iterations. Appendix B proves the convergence of the map to a stable fixed point, while Appendix C is a miscellanea of proofs of various statements used in the paper.

1. THE APPROACH OF RANDOM TRANSFER MATRICES

The Hamiltonian H_N of N spins $\{\sigma_i = \pm 1\}_{i=1,\dots,N}$ on a one-dimensional lattice with periodic boundary conditions $\{\sigma_{i+N} = \sigma_i\}$ is of the type

$$-\beta H_N = \sum_{i=1}^{N} J_i \sigma_i \sigma_{i+1} + \sum_{i=1}^{N} h_i \sigma_i$$

where β is the inverse temperature, and where the N couples of nearest-neighbor couplings J_i and external magnetic fields h_i are independent identically distributed random variables with probability measure $\mu(J_i, h_i)$. The previous definitons of the J_i and the h_i include a factor β with respect to the usual ones, in order to simplify the notation.

The partition function $Z_N = \mathrm{Tr}\exp(-\beta H_N)$ depends on the particular realization $\{J_i, h_i\}$, and it can be written as the trace of the product of N random transfer matrices:

$$Z_N = \mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_i \tag{1.1}$$

with

$$\mathbf{A}_i = \begin{pmatrix} e^{J_i + h_i} & e^{-J_i + h_i} \\ e^{-J_i - h_i} & e^{J_i - h_i} \end{pmatrix} \tag{1.2}$$

An important physical quantity, the *quenched* free energy f, is directly related to $\lambda_1 = -\beta f$, the maximum Lyapunov exponent, defined as

$$\lambda_1 = \lim_{N \to \infty} \frac{1}{N} \overline{\ln \mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_i} \tag{1.3}$$

where $\bar{\cdot}$ represents the average over the disorder probability distribution $\prod_i \mu(J_i, h_i)$. To simplify the notation, we often omit the subscript i, and J, h indicate the generic random variables J_i, h_i.

The difficulty in computing λ_1 is due to the presence of the logarithm function inside the average. A simple but rather poor upper bound of the maximum Lyapunov exponent can be obtained by moving the logarithm outside the disorder average. This is the so-called *annealed* average

$$L = \lim_{N \to \infty} \frac{1}{N} \ln \overline{\mathrm{Tr} \prod_{i=1}^{N} \mathbf{A}_i} \tag{1.4}$$

Calling $\mathbf{A}$ the generic random matrix of type (1.2) depending on J and h, if $\mu(J, h)$ is irreducible,[3] and with the request

$$\overline{\ln^+ \|\mathbf{A}\|} = \overline{\ln[e^J \cosh h + (e^{2J} \sinh^2 h + e^{-2J})^{1/2}]} < \infty \tag{1.5}$$

where we have used the eigenvalue of A with maximum modulus as matrix norm, the Furstenberg theorem[4] assures that for $\mu(J, h)$-almost all (μ-a.a.) the possible products (a part a set of zero probability measure) one has

$$\lim_{N \to \infty} \frac{1}{N} \ln \operatorname{Tr} \prod_{i=1}^{N} \mathbf{A}_i = \lambda_1 \qquad \mu\text{-a.a.} \tag{1.6}$$

In statistical physics this property is often called the *self-average* of the maximum Lyapunov exponent. Moreover, using the multiplicative ergodic theorem of Oseledec[5] with the previous requests, one can introduce another self-averaging Lyapunov exponent, $\lambda_2 \leqslant \lambda_1$, which can be defined by the relation

$$\lambda_1 + \lambda_2 = \lim_{N \to \infty} \frac{1}{N} \ln \overline{\left| \det \prod_{i=1}^{N} \mathbf{A}_i \right|} = \overline{\ln(2 \sinh 2 |J|)} \tag{1.7}$$

In order to have λ_2 finite, we will always require that the right-hand side of (1.5) is finite, together with the requests of the Furstenberg theorem, i.e., that $\mu(J, h)$ is irreducible and (1.5). It is thus sufficient that the J distribution has no delta function in $J = 0$. Notice that (1.5) implies that $\mu(J, h)$ tends to 0 faster than $(|J| \cdot |h|)^{-2}$ in the limit $|J| \to \infty$ or $|h| \to \infty$.

2. THE MAP

In this section we derive a deterministic map of the couplings J_i and the magnetic fields h_i of the one-dimensional random Ising model. To achieve our goal, the partition function Z_N should be written as

$$Z_N = 2^N \left\langle \prod_{i=1}^{N} e^{J_i \sigma_i \sigma_{i+1} + h_i \sigma_i} \right\rangle_{\{\sigma_i\}}$$

where the average $\langle \cdots \rangle_{\{\sigma_i\}}$ is performed over the 2^N spin configurations $\{\sigma_i\}$. Let us now introduce a second set of spinlike variables $\{\tau_i\}$ so that

$$e^{J_i \sigma_i \sigma_{i+1}} = e^{-|J_i|} \langle e^{\tau_{i+1} \mathscr{C}_i (\xi_i \sigma_i + \sigma_{i+1})} \rangle_{\tau_{i+1}} \tag{2.1}$$

[3] There is no nontrivial subspace $\mathbf{V}$ of $\mathbb{R}^2$ such that $\mathbf{A}(\mathbf{V}) = \mathbf{V}$ for $\mu(J, h)$-almost all $\mathbf{A}$.

where

$$\begin{cases} \mathscr{C}_i = \dfrac{1}{2} \cosh^{-1}(e^{2|J_i|}) \\ \xi_i = \dfrac{J_i}{|J_i|} \end{cases} \qquad \forall i = 1, ..., N \tag{2.2}$$

and therefor $\mathscr{C}_i$ is a nonnegative real number. By means of (2.1), and performing the average over $\{\sigma_i\}$, we find for the partition function Z_N

$$Z_N = 2^N \left\langle \prod_{i=1}^{N} e^{-|J_i|} \cosh(\mathscr{C}_{i-1}\tau_i + \mathscr{C}_i \xi_i \tau_{i+1} + h_i) \right\rangle_{\{\tau_i\}} \tag{2.3}$$

We have thus obtained a partition function which is linear in the spin variables instead of being bilinear. The partition function (2.3) can again be calculated as the trace of a product of appropriate transfer matrices, since it is always possible to find a set of constants $\{W_i, X_i, Y_i, Z_i\}$ which satisfy the identity

$$\cosh(\mathscr{C}_{i-1}\tau_i + \mathscr{C}_i \xi_i \tau_{i+1} + h_i) = e^{W_i + X_i\tau_i + Y_i\tau_{i+1} + Z_i\tau_i\tau_{i+1}} \tag{2.4}$$

Inserting (2.4) in (2.3), it follows that

$$Z_N = \exp\left(\sum_{i=1}^{N} (W_i - |J_i|) \right) \operatorname{Tr} \prod_{i=1}^{N} \mathbf{A}_i^{(1)} \tag{2.5}$$

where $\mathbf{A}_i^{(1)}$ are the random transfer matrices of a new one-dimensional Ising model with nearest-neighbor couplings $\{J_i^{(1)}\}$ and external magnetic fields $\{h_i^{(1)}\}$ given by the deterministic map $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{R}^N$:

$$\begin{cases} J_i^{(1)} = Z_i = \xi_i \langle xyg(x\mathscr{C}_{i-1} + y\mathscr{C}_i + h_i) \rangle_{x,y} \\ h_i^{(1)} = X_i + Y_{i-1} \\ \quad\;\; = \langle xg(x\mathscr{C}_{i-1} + y\mathscr{C}_i + h_i) + x\xi_{i-1}\, g(x\mathscr{C}_{i-1} + y\mathscr{C}_{i-2} + h_{i-1}) \rangle_{x,y} \end{cases} \tag{2.6}$$

where $g(z) \equiv \ln \cosh z$, and $x = \pm 1$ and $y = \pm 1$ with equal probability are two auxiliary random variables introduced only to obtain a more compact expression of the formulas. Let us stress that the variable W_i does not appear in the map. Although possible, it is not convenient to compute it in order to obtain a formal expression of the Lyapunov exponents, as we discuss in the next section.

It is important to notice that the new coupling $J_i^{(1)}$ depends on J_{i-1}, J_i (through $\mathscr{C}_{i-1}, \mathscr{C}_i$), while the new field $h_i^{(1)}$ is a function of J_{i-2}, J_{i-1}, J_i, h_{i-1}, and h_i. In other terms the quenched variables of the transformed system on a site depend on the quenched variables of the original systems in the same site and in two neighbor sites. As a consequence they are no longer independent random variables. It is also easy to check that $J_i^{(1)}$ has the same sign of J_i.

The original system with couplings and fields $\{J_i, h_i\}$ can be thought of as the initial step $n=0$, and its various quantities are indifferently written with or without the 0-superscript, i.e., $\lambda_1^{(0)} \equiv \lambda_1$, $J_i^{(0)} \equiv J_i$, $W_i^{(0)} \equiv W_i$, and so on.

3. FORMAL EXPRESSION OF THE LYAPUNOV EXPONENTS

Relations (1.1) and (2.5) show that the two matrices $(\prod_{i=1}^N \mathbf{A}_i)$ and $(\prod_{i=1}^N \mathbf{A}_i^{(1)} \exp(W_i - J_i))$ have equal trace. A direct inspection shows that they also have equal determinant, so that they are similar. As a consequence, taking the limit $N \to \infty$, one can directly relate λ_1 and λ_2 to $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ (the Lyapunov exponents of the product of matrices $\mathbf{A}_i^{(1)}$):

$$\begin{cases} \lambda_1 + \lambda_2 = \lambda_1^{(1)} + \lambda_2^{(1)} + 2(\bar{W} - \overline{|J|}) \\ \lambda_1 - \lambda_2 = \lambda_1^{(1)} - \lambda_2^{(1)} \end{cases} \tag{3.1}$$

Let us notice that every mean value, such as, for example, $\overline{|J_i|}$, does not depend on the site i, which is omitted. Moreover, $\lambda_1^{(1)}$ and $\lambda_2^{(1)}$ are well-defined self-averaging quantities, since $\overline{|J|} < \infty$ and $\bar{W} < \infty$ [see (1.5) and Appendix C, proof 4]. Equation (3.1) can be simplified by recalling the expression (1.7), which is valid both for $n=0$ and $n=1$, so that

$$\lambda_1^{(n)} + \lambda_2^{(n)} = \overline{\ln(2 \sinh 2\, |J^{(n)}|)} \tag{3.2}$$

We can now iterate the map n times, using again (3.2) and the fact that the difference of the Lyapunov exponents is invariant, to obtain the expression

$$\begin{cases} \lambda_1 = \dfrac{1}{2} \overline{\ln \dfrac{\sinh 2\, |J|}{\sinh 2\, |J^{(n)}|}} + \lambda_1^{(n)} \\ \lambda_2 = \dfrac{1}{2} \overline{\ln \dfrac{\sinh 2\, |J|}{\sinh 2\, |J^{(n)}|}} + \lambda_2^{(n)} \end{cases} \tag{3.3}$$

where $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ are the Lyapunov exponents of the system after n iterations of the map (2.6), and we have assumed that $\overline{|J^{(n)}|} < \infty$ and $\overline{W^{(n)}} < \infty$ (see Appendix C, proofs 1 and 4). Moreover, they are still self-averaging, as shown in Appendix C, proof 5. The coupling $J^{(n)}$ is a function of $n+1$ initial couplings and n initial magnetic fields, while the magnetic field $h^{(n)}$ depends on $n+2$ couplings and $n+1$ fields.

Let us take the limit $n \to \infty$: the sequence $\overline{\ln(\sinh 2\,|J^{(n)}|)}$ is nonincreasing in n (see Appendix C, proof 8), and so it is convergent, since $\lambda_1^{(n)} \geqslant 0$. As a consequence, $\lim_{n \to \infty} \lambda_1^{(n)}$ exists, and it turns out to be (proof 10)

$$\lim_{n \to \infty} \lambda_1^{(n)} = \lim_{n \to \infty} \overline{\ln(2 \cosh J^{(n)})} \tag{3.4}$$

The expression of the Lyapunov exponents is thus

$$\begin{cases} \lambda_1 = \frac{1}{2}\,\overline{\ln(2 \sinh 2\,|J|)} - \frac{1}{2} \ln \tanh J^* \\ \lambda_2 = \frac{1}{2}\,\overline{\ln(2 \sinh 2\,|J|)} + \frac{1}{2} \ln \tanh J^* \end{cases} \tag{3.5}$$

with $J^* > 0$ such that

$$\ln \tanh J^* = \lim_{n \to \infty} \overline{\ln\ \tanh |J^{(n)}|} \tag{3.6}$$

Unfortunately, we are not able to give the form of J^* in terms of known functions. However, from a numerical point of view this result allows one to give a very good estimate of the Lyapunov spectrum by a truncation of the sequence at finite n. In the next section we shall show that these estimates are also a lower and an upper bound of λ_1 and λ_2, respectively.

4. RIGOROUS BOUNDS OF THE LYAPUNOV EXPONENTS

In the previous section we have found a formal expression of the Lyapunov exponents (3.5) in terms of the limit of the sequence $\overline{\ln \tanh |J^{(n)}|}$, (3.6). Indeed it is sufficient to stop at finite n to get rigorous bounds for λ_1 and λ_2. In fact the sequence $\overline{\ln \tanh |J^{(n)}|}$ is nonincreasing in n (see Appendix C, proof 6), so that (3.5) implies

$$\begin{cases} \lambda_1 \geqslant \frac{1}{2}\,\overline{\ln(2 \sinh 2\,|J|)} - \frac{1}{2}\,\overline{\ln \tanh |J^{(n)}|} \\ \lambda_2 \leqslant \frac{1}{2}\,\overline{\ln(2 \sinh 2\,|J|)} + \frac{1}{2}\,\overline{\ln \tanh |J^{(n)}|} \end{cases} \tag{4.1}$$

The previous bounds of the Lyapunov exponents converge monotonously to λ_1 and λ_2 in the limit $n \to \infty$.

In order to give rigorous bounds for the Lyapunov exponents in the opposite direction, let us apply the map (2.6) $n-1$ times. At the nth step we modify the relation (2.1), changing two signs as follows:

$$\exp(J_i^{(n-1)} \sigma_i \sigma_{i+1}) = (\exp |J_i^{(n-1)}|) \langle \exp[\tau_{i+1} \mathscr{C}_i^{(n-1)}(\xi_i \sigma_i - \sigma_{i+1})] \rangle_{\tau_{i+1}}$$

so that $\tilde{\mathscr{C}}_i^{(n)}$ is an imaginary quantity:

$$\tilde{\mathscr{C}}_i^{(n-1)} = \tfrac{1}{2}\cosh^{-1}[\exp(-2\,|J_i^{(n-1)}|)]$$

Going on as before, we find a new deterministic map:

$$\begin{cases} \tilde{J}_i^{(n)} = -\xi_i \langle xyg(x\tilde{\mathscr{C}}_{i-1}^{(n-1)} + y\tilde{\mathscr{C}}_i^{(n-1)} + h_i^{(n-1)})\rangle_{x,y} \\ \tilde{h}_i^{(n)} = \langle -xg(x\tilde{\mathscr{C}}_{i-1}^{(n-1)} + y\tilde{\mathscr{C}}_i^{(n-1)} + h_i^{(n-1)}) \\ \qquad + x\xi_{i-1}\, g(x\tilde{\mathscr{C}}_{i-1}^{(n-1)} + y\tilde{\mathscr{C}}_{i-2}^{(n-1)} + h_{i-1}^{(n-1)})\rangle_{x,y} \end{cases} \tag{4.2}$$

where $\{\tilde{J}_i^{(n)}\}$ and $\{\tilde{h}_i^{(n)}\}$ are, respectively, the couplings and the magnetic fields of a new one-dimensional Ising model with Lyapunov exponents $\{\tilde{\lambda}_1^{(n)}\}$ and $\{\tilde{\lambda}_2^{(n)}\}$, related to λ_1 and λ_2 by

$$\begin{cases} \lambda_1 = \dfrac{1}{2}\ln\overline{\dfrac{\sinh 2\,|J|}{\sinh 2\,|\tilde{J}^{(n)}|}} + \tilde{\lambda}_1^{(n)} \\ \lambda_2 = \dfrac{1}{2}\ln\overline{\dfrac{\sinh 2\,|J|}{\sinh 2\,|\tilde{J}^{(n)}|}} + \tilde{\lambda}_2^{(n)} \end{cases}$$

It is easy to show that every $\tilde{h}_i^{(n)}$ is an imaginary quantity [the map (4.2) is an application of $\mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N \times \mathbb{I}^N$], and therefore the following inequalities hold:

$$\begin{cases} \tilde{\lambda}_1^{(n)} \geqslant \overline{\ln(2\cosh|\tilde{J}^{(n)}|)} \\ \tilde{\lambda}_2^{(n)} \leqslant \overline{\ln(2\cosh|\tilde{J}^{(n)}|)} \end{cases}$$

that is,

$$\begin{cases} \lambda_1 \leqslant \tfrac{1}{2}\,\overline{\ln(2\sinh 2\,|J|)} - \tfrac{1}{2}\,\overline{\ln\tanh|\tilde{J}^{(n)}|} \\ \lambda_2 \geqslant \tfrac{1}{2}\,\overline{\ln(2\sinh 2\,|J|)} + \tfrac{1}{2}\,\overline{\ln\tanh|\tilde{J}^{(n)}|} \end{cases} \tag{4.3}$$

Combining these results with (4.1), we have obtained upper and lower bounds of the Lyapunov spectrum. In Appendix C, proof 11, we show that also these new bounds (4.3), as well the previous ones (4.1), converge to the Lyapunov exponents λ_1 and λ_2 in the limit $n \to \infty$, that is,

$$\lim_{n\to\infty} \overline{\ln\tanh|\tilde{J}^{(n)}|} = \ln\tanh J^* \tag{4.4}$$

The relations (4.1) and (4.3) represent a practical tool for calculating the Lyapunov exponents of a one-dimensional random Ising model when the couplings and the magnetic fields are discrete random variables with few possible values. As an example, we have considered the case with constant couplings $J_i = J$ and binomial magnetic fields $h_i = H \pm h$ ($H \geqslant 0$, $h > 0$, where the signs $\pm$ are chosen with equal probability). In Fig. 1 we plot the

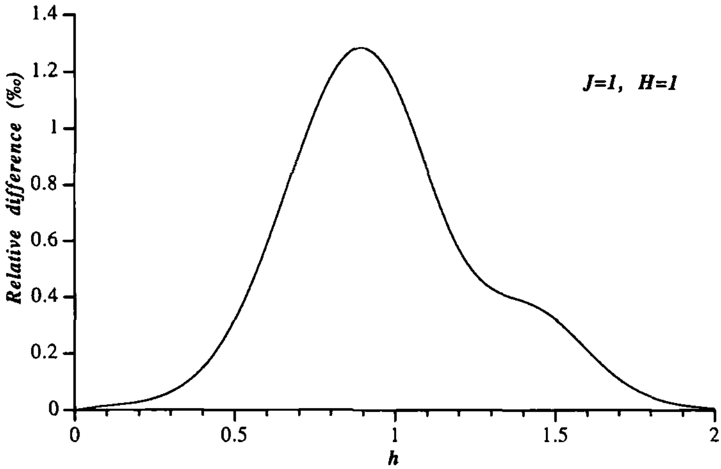

Fig. 1. Random Ising model with constant couplings $J_i = 1$ and magnetic fields $h_i = 1 \pm h$ with equal probability of $\pm$: relative difference of the upper and lower bounds of the maximum Lyapunov exponent λ_1 after $n = 15$ iterations, as a function of h.

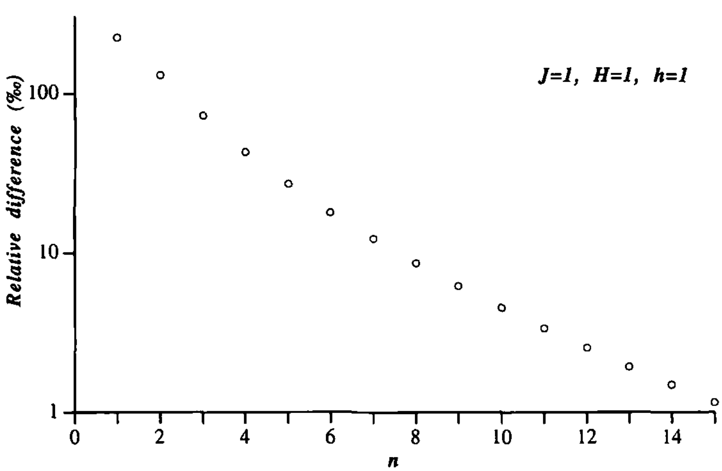

Fig. 2. Random Ising model with constant couplings $J_i = 1$ and magnetic fields $h_i = 1 \pm 1$ with equal probability of $\pm$: relative difference of the upper and lower bounds of the maximum Lyapunov exponent λ_1 as a function of n, the number of iterations (log-linear scale).

relative difference of the two bounds (4.1) and (4.3) of the maximum Lyapunov exponent λ_1 as a function of h for $n=15$, $J=1$, and $H=1$. Notice that after $n=15$ iterations the relative error is, in the worst case, about 10^{-3}. Our results allow one to get estimates which have a precision higher than standard Monte Carlo simulations.[2] Moreover, we are able to find rigorously the interval where the Lyapunov exponents fall. Finally, let us stress that the convergence in n is quite fast; see, for instance, Fig. 2 for the case $J=1$, $H=1$, and $h=1$, where the relative error exponentially decreases with n.

APPENDIX A

In this paper we have always performed the thermodynamic limit $N \to \infty$ before the limit $n \to \infty$, where n is the number of iterations of the map. The exchange of those limits allows us to characterize the variable J^* introduced in Section 3.

In this order of things we have to consider the system with a finite number N of sites. Calling $\lambda_{1,N}$, $\lambda_{2,N}$ the Lyapunov exponents in this context, and $\lambda_{1,N}^{(n)}$, $\lambda_{2,N}^{(n)}$ the same quantities after n iterations of the map, we find the following relations:

$$\begin{cases} \lambda_{1,N} = \dfrac{1}{2} \overline{\ln \dfrac{\sinh 2\,|J|}{\sinh 2\,|J^{(n)}|}} + \lambda_{1,N}^{(n)} \\ \lambda_{2,N} = \dfrac{1}{2} \overline{\ln \dfrac{\sinh 2\,|J|}{\sinh 2\,|J^{(n)}|}} + \lambda_{2,N}^{(n)} \end{cases} \tag{A.1}$$

where the averages involve the N initial couples $\{J_i, h_i\}$ if $n \geqslant N-2$. Similar arguments to those of Section 3 assure that these relations are meaningful.

Let us restrict the ensemble of initial disorder realizations as follows:

$$\begin{cases} |J_i| < \sup \omega_N \\ |h_i| < \sup \omega_N \end{cases} \qquad \forall i = 1,..., N$$

where ω_N is defined by the following relation:

$$\int_{-\omega_N}^{+\omega_N} dJ \int_{-\omega_N}^{+\omega_N} dh\, \mu(J, h) \leqslant 1 - \frac{1}{N^{1+\rho}}, \qquad \rho > 0$$

in such a way that the restricted ensemble of N independent couples $\{J_i, h_i\}$ has probability one for $N \to \infty$. Moreover, it is easy to show that the Lyapunov exponents λ_1 and λ_2 do not vary with this restriction, as well as $\overline{\ln \sinh 2|J|}$, in the limit $N \to \infty$.

For all the initial disorder realizations of the new ensemble, the map (2.6) converges for $n \to \infty$ to a fixed point (see Appendix B) of type

$$\begin{cases} \lim_{n\to\infty} J_i^{(n)} = \xi_i J^*_{(N)} \\ \lim_{n\to\infty} h_i^{(n)} = 0 \end{cases} \qquad \forall i = 1,..., N$$

with $J^*_{(N)} \geqslant 0$ depending on the intial values $\{J_i, h_i\}_{i=1,...,N}$. The fixed point corresponds to a product of random independent matrices of the form

$$\mathbf{A}_i^{(\infty)} = \begin{pmatrix} e^{\xi_i J^*_{(N)}} & e^{-\xi_i J^*_{(N)}} \\ e^{-\xi_i J^*_{(N)}} & e^{\xi_i J^*_{(N)}} \end{pmatrix}$$

By means of this, in the limit $n \to \infty$ the relations (A.1) become

$$\begin{cases} \lambda^\circ_{1,N} = \dfrac{1}{2}\,\overline{\ln(2\sinh 2|J|)}^\circ - \dfrac{1}{2}\,\overline{\ln\tanh J^*_{(N)}}^\circ + \dfrac{1}{N}\,\overline{\ln(1 \pm \tanh^N J^*_{(N)})}^\circ \\ \lambda^\circ_{2,N} = \dfrac{1}{2}\,\overline{\ln(2\sinh 2|J|)}^\circ + \dfrac{1}{2}\,\overline{\ln\tanh J^*_{(N)}}^\circ - \dfrac{1}{N}\,\overline{\ln(1 \pm \tanh^N J^*_{(N)})}^\circ \end{cases} \tag{A.2}$$

were the sign $\pm$ in the last term of both expressions is equal to $\prod_{i=1}^N \xi_i$, and the circles recall the restriction of the ensemble.

Now we perform the thermodynamic limit ($N \to \infty$). As previously discussed, $\lambda^\circ_{1,N}$ and $\lambda^\circ_{2,N}$ converge to the Lyapunov exponents λ_1 and λ_2, while $\overline{\ln\sinh 2|J|}^\circ$ tends to $\overline{\ln\sinh 2|J|}$. The results of Appendix C can be easily estended to the case with finite N and restricted ensemble. That allows us to show that the last term of both the expressions of $\lambda_{2,N}$ and $\lambda_{2,N}$ vanishes in the thermodynamic limit, so that

$$\lim_{N\to\infty} \overline{\ln\tanh J^*_{(N)}}^\circ = \ln\tanh J^* \tag{A.3}$$

where J^* is the coupling defined at the end of Section 3. It is also simple to see that $\lim_{N\to\infty} \ln\tanh J^*_{(N)}$ has necessarily the same value for a set of initial disorder realizations $\{J_i, h_i\}_{i=1,...,\infty}$ of probability one. In conclusion, we have

$$J^* = \lim_{N\to\infty} J^*_{(N)} = \lim_{N\to\infty}\lim_{n\to\infty} J_i^{(n)} \qquad \forall i, \quad \mu\text{-a.a.} \tag{A.4}$$

This result is important from a theoretical point of view, although it is rather too slowly convergent to be useful in a numerical calculation in contrast with (3.6) and with the upper and lower bounds obtained in Section 4.

APPENDIX B

In this appendix we prove the convergence of the map (2.6) for finite N sites. It is more convenient to consider the variables $\{\mathscr{C}_i^{(n)}\}$ instead of the couplings $\{J_i^{(n)}\}$ of the nth system (i.e., the system after applied the map n times). We recall that these variables are related by

$$\cosh 2\mathscr{C}_i^{(n)} = e^{2|J_i^{(n)}|}, \qquad \forall i = 1,..., N$$

and that all the $\mathscr{C}_i^{(n)}$ are nonnegative real numbers. Let us briefly give the outlook of this appendix: we first prove the convergence in n of $\max_i \mathscr{C}_i^{(n)}$ and $\min_i \mathscr{C}_i^{(n)}$ to the same value $\mathscr{C}^*_{(N)}$, which ensures the convergence of every single $\mathscr{C}_i^{(n)}$ to $\mathscr{C}^*_{(N)}$. Then we consider the absolute magnetic field $|h_i^{(n)}|$, which tends to 0 in the limit $n \to \infty$. Let us stress that we always consider an initial disorder realization $\{J_i, h_i\}$ such that

$$\max_i \mathscr{C}_i^{(n)} < \infty, \qquad \max_i |h_i^{(n)}| < \infty$$

From (2.6) one can derive the iterated relation

$$\cosh 2\mathscr{C}_i^{(n+1)} = \left[\frac{\cosh 2h_i^{(n)} + \cosh(2\mathscr{C}_i^{(n)} + 2\mathscr{C}_{i-1}^{(n)})}{\cosh 2h_i^{(n)} + \cosh(2\mathscr{C}_i^{(n)} - 2\mathscr{C}_{i-1}^{(n)})}\right]^{1/2} \tag{B1}$$

and $\{J_i^{(0)} \equiv J_i,\ h_i^{(0)} \equiv h_i\}$ represent the couplings and the magnetic fields of the initial system.

From (B1) the following inequalities hold:

$$\begin{aligned}\cosh 2\mathscr{C}_i^{(n+1)} &\leqslant \left[\frac{1 + \cosh(2\mathscr{C}_i^{(n)} + 2\mathscr{C}_{i-1}^{(n)})}{1 + \cosh(2\mathscr{C}_i^{(n)} - 2\mathscr{C}_{i-1}^{(n)})}\right]^{1/2} \\ &= \frac{\cosh(\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)})}{\cosh(\mathscr{C}_i^{(n)} - \mathscr{C}_{i-1}^{(n)})} \leqslant \cosh(\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)})\end{aligned} \tag{B2}$$

that is,

$$\mathscr{C}_i^{(n+1)} \leqslant \frac{\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)}}{2} \tag{B3}$$

Performing the maximum over i

$$\max_i \mathscr{C}_i^{(n+1)} \leqslant \max_i \mathscr{C}_i^{(n)}$$

Since $\max_i \mathscr{C}_i^{(n)} \geqslant 0$, the previous relation proves the convergence of $\max_i \mathscr{C}_i^{(n)}$ to a nonnegative real number, say $\mathscr{C}^*_{(N)} < \infty$.

In order to show the convergence of $\min_i \mathscr{C}_i^{(n)}$, let us suppose that

$$\min_i \mathscr{C}_i^{(n)} = \mathscr{C}^*_{(N)} - \delta_n \tag{B4}$$

with $\delta_n > 0$. Since $\lim_{n\to\infty} \max_i \mathscr{C}_i^{(n)} = \mathscr{C}^*_{(N)}$, there exists $\varepsilon_n > 0$ such that

$$\max_i \mathscr{C}_i^{(n')} < \mathscr{C}^*_{(N)} + \varepsilon_n, \qquad \forall n' \geqslant n \tag{B5}$$

Iterating (B3) $N-1$ times, one has

$$\mathscr{C}_i^{(n+N-1)} \leqslant \frac{1}{2^{N-1}} \sum_{k=0}^{N-1} \binom{N-1}{k} \mathscr{C}_{i-k}^{(n)}$$

where the index k runs all over the N sites. Using (B4) for a site where the minimum of $\mathscr{C}_i^{(n)}$ is realized, and (B5) for the remaining sites, and also performing the maximum over i, we find

$$\max_i \mathscr{C}_i^{(n+N-1)} < \mathscr{C}^*_{(N)} + \varepsilon_n - \frac{1}{2^{N-1}} (\varepsilon_n + \delta_n)$$

Keeping in mind that $\max_i \mathscr{C}_i^{(n+N-1)} \geqslant \mathscr{C}^*_{(N)}$, we have that the following inequality holds:

$$\delta_n < (2^{N-1} - 1)\, \varepsilon_n$$

which proves that $\lim_{n\to\infty} \min_i \mathscr{C}_i^{(n)} = \mathscr{C}^*_{(N)}$, and therefore

$$\lim_{n\to\infty} \mathscr{C}_i^{(n)} = \mathscr{C}^*_{(N)}, \qquad \forall i = 1,..., N$$

Recalling that every $J_i^{(n)}$ keeps the initial sign ξ_i, it immdiately follows that

$$\lim_{n\to\infty} J_i^{(n)} = \xi_i J^*_{(N)}, \qquad \forall i = 1,..., N$$

where $J^*_{(N)}$ is a nonnegative real number with

$$\cosh 2\mathscr{C}^*_{(N)} = e^{2J^*_{(N)}}$$

We have now to show that every magnetic field $h_i^{(n)}$ converges to 0 in the limit $n \to \infty$. In this order of things, from (B1) we obtain

$$\cosh^2 h_i^{(n)} \leqslant \frac{\sinh 2\mathscr{C}_i^{(n)} \sinh 2\mathscr{C}_{i-1}^{(n)}}{\sinh^2 2\mathscr{C}_i^{(n+1)}}$$

If $\mathscr{C}^*_{(N)} > 0$, the previous inequality is sufficient to reach our goal.

In the other case $\mathscr{C}^*_{(N)} = 0$, recalling (B5), from the second equation of the map (2.6) we can get

$$\begin{aligned} |h_i^{(n+1)}| < \tfrac{1}{4} \ln\{ & [\cosh(|h_i^{(n)}| + 2\varepsilon_n) \cosh(|h_i^{(n)}| + \varepsilon_n) \\ & \times \cosh(|h_{i-1}^{(n)}| + 2\varepsilon_n) \cosh(|h_{i-1}^{(n)}| + \varepsilon_n)] \\ & \times [\cosh(|h_i^{(n)}| - 2\varepsilon_n) \cosh(|h_i^{(n)}| - \varepsilon_n) \\ & \times \cosh(|h_{i-1}^{(n)}| - 2\varepsilon_n) \cosh(|h_{i-1}^{(n)}| - \varepsilon_n)]^{-1}\} \end{aligned}$$

Since

$$\frac{\cosh(|h_i^{(n)}| + \varepsilon_n)}{\cosh(|h_i^{(n)}| - \varepsilon_n)} \leqslant e^{2\varepsilon_n}$$

and similarly for the other factors, it immediately follows that

$$|h_i^{(n+1)}| < 3\varepsilon_n$$

which proves that

$$\lim_{n \to \infty} |h_i^{(n)}| = 0$$

APPENDIX C

In this appendix we collect the proofs of several statements used in the text. Let us stress that the average of a quantity, say $\overline{\phi_i}$, does not depend on the site i, which is often omitted. Moreover, we recall that the thermodynamic limit ($N \to \infty$) already has been performed. As a consequence at the nth step the average is performed over $n+2$ initial nearest-neighbor couplings and $n+1$ magnetic fields.

1. $\overline{|J^{(n+1)}|} \leqslant \overline{|J^{(n)}|} < \infty$. Recalling (2.2) and (B3), one has

$$|J_i^{(n+1)}| \leqslant \tfrac{1}{2} \ln \cosh(\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)})$$

Using the convexity of the function $\ln\cosh(\cdot)$, we obtain

$$|J_i^{(n+1)}| \leqslant \tfrac{1}{4} \ln\cosh(2\mathscr{C}_i^{(n)}) + \tfrac{1}{4} \ln\cosh(2\mathscr{C}_{i-1}^{(n)}) = \tfrac{1}{2}(|J_i^{(n)}| + |J_{i-1}^{(n)}|) \quad \text{(C1)}$$

Averaging the previous expression, it immediately follows that

$$\overline{|J^{(n+1)}|} \leqslant \overline{|J^{(n)}|}$$

The quantity $\overline{|J^{(n)}|}$ is finite, since $\overline{|J^{(0)}|} < \infty$, because of condition (1.5).

2. $\overline{\mathscr{C}^{(n+1)}} \leqslant \overline{\mathscr{C}^{(n)}} < \infty$. Averaging (B3), we prove that $\overline{\mathscr{C}^{(n)}}$ is a nonincreasing quantity in n. Moreover,

$$\ln \cosh 2\overline{\mathscr{C}^{(n)}} \leqslant \overline{\ln \cosh 2\mathscr{C}^{(n)}} = 2\,\overline{|J^{(n)}|}$$

which ensures that $\overline{\mathscr{C}^{(n)}}$ is finite (see proof 1).

3. $\overline{|h^{(n)}|} < \infty$. From the expression of $h_i^{(n+1)}$ in the map (2.6) it follows that

$$\overline{|h_i^{(n+1)}|}$$

$$\leqslant \frac{1}{4} \overline{\ln \frac{\cosh(2\,|h_i^{(n)}| + 2\mathscr{C}_{i-1}^{(n)}) + \cosh 2\mathscr{C}_i^{(n)}}{\cosh(2\,|h_i^{(n)}| - 2\mathscr{C}_{i-1}^{(n)}) + \cosh 2\mathscr{C}_i^{(n)}} \cdot \frac{\cosh(2\,|h_i^{(n)}| + 2\mathscr{C}_i^{(n)}) + \cosh 2\mathscr{C}_{i-1}^{(n)}}{\cosh(2\,|h_i^{(n)}| - 2\mathscr{C}_i^{(n)}) + \cosh 2\mathscr{C}_{i-1}^{(n)}}}$$

Since $[\cosh(a+b)+c]/[\cosh(a-b)+c] \leqslant e^{2a}$ for $a, b, c \geqslant 0$, one has (see also proof 2)

$$\overline{|h^{(n+1)}|} \leqslant 2\overline{\mathscr{C}^{(n)}} < \infty$$

4. $\overline{W^{(n)}} < \infty$. From (2.4) one finds the expression of $\overline{W^{(n)}}$:

$$\overline{W_{(i)}^{(n)}} = \overline{\langle \ln \cosh(h_i^{(n)} + x\mathscr{C}_i^{(n)} + y\mathscr{C}_{i-1}^{(n)}) \rangle_{x,y}}$$

Since $\ln \cosh a \leqslant |a|$, and from proofs 2 and 3, we have

$$\overline{W^{(n)}} \leqslant \overline{|h^{(n)}|} + 2\overline{\mathscr{C}^{(n)}} < \infty$$

5. $\lambda_1^{(n)}$ and $\lambda_2^{(n)}$ self-average. We suppose that this property holds at the step $n-1$. It thus suffices to prove that the maximum Lyapunov exponent of each disorder realization [i.e., (1.3) without average] is a nonnegative and nonincreasing quantity in n. The first statement is easy to see, while for the second we have to show that $\sum_{i=1}^{N} (W_i^{(n-1)} - |J_i^{(n-1)}|)$ is nonnegative. From the expression of $W_i^{(n-1)}$ (see proof 4), and using the convexity of the logarithm, we have

$$W_i^{(n-1)} \geqslant \tfrac{1}{2} \ln(\tfrac{1}{2} \cosh 2\mathscr{C}_i^{(n-1)} + \tfrac{1}{2} \cosh 2\mathscr{C}_{i-1}^{(n-1)}) \geqslant \tfrac{1}{2}(|J_i^{(n-1)}| + |J_{i-1}^{(n-1)}|)$$

which concludes the proof.

6. $\overline{\ln \tanh |J^{(n+1)}|} \leqslant \overline{\ln \tanh |J^{(n)}|}$. We first notice that

$$\tanh |J_i^{(n)}| = \frac{\cosh 2\mathscr{C}_i^{(n)} - 1}{\cosh 2\mathscr{C}_i^{(n)} + 1} = \tanh^2 \mathscr{C}_i^{(n)}$$

Recalling the first part of (B2), we have that the following inequality holds:

$$\tanh^2 \mathscr{C}_i^{(n+1)} \leqslant \frac{\cosh(\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)}) - \cosh(\mathscr{C}_i^{(n)} - \mathscr{C}_{i-1}^{(n)})}{\cosh(\mathscr{C}_i^{(n)} + \mathscr{C}_{i-1}^{(n)}) + \cosh(\mathscr{C}_i^{(n)} - \mathscr{C}_{i-1}^{(n)})} = \tanh \mathscr{C}_i^{(n)} \tanh \mathscr{C}_{i-1}^{(n)}$$

Computing the logarithm and averaging, we obtain

$$\overline{\ln \tanh |J^{(n+1)}|} \leqslant \overline{\ln \tanh |J^{(n)}|}$$

7. $\overline{\ln \cosh J^{(n+1)}} \leqslant \overline{\ln \cosh J^{(n)}}$. Taking into account (C1) and the convexity of $\ln \cosh(\cdot)$, one has

$$\overline{\ln \cosh J^{(n+1)}} \leqslant \overline{\ln \cosh(\tfrac{1}{2} |J_i^{(n)}| + \tfrac{1}{2} |J_{i-1}^{(n)}|)} \leqslant \overline{\ln \cosh J_i^{(n)}}$$

8. $\overline{\ln \sinh 2 |J^{(n+1)}|} \leqslant \overline{\ln \sinh 2 |J^{(n)}|}$. Since

$$\overline{\ln \sinh 2 |J^{(n)}|} = \ln 2 + \overline{\ln \tanh |J^{(n)}|} + 2\, \overline{\ln \cosh J^{(n)}}$$

from proofs 6 and 7, it follows that $\overline{\ln \sinh 2 |J^{(n)}|}$ is a nonincreasing quantity in n.

9. $\lim_{n\to\infty} \overline{|h^{(n)}|} = 0$. From the first relation of the map (2.6) the following inequality holds:

$$\cosh h_i^{(n)} \leqslant \frac{\sinh 2\mathscr{C}_i^{(n)} \sinh 2\mathscr{C}_{i-1}^{(n)}}{\sinh^2 2\mathscr{C}_i^{(n+1)}}$$

The previous relation reads:

$$\begin{aligned}\ln \cosh \overline{h^{(n)}} &\leqslant \overline{\ln \sinh 2\mathscr{C}^{(n)}} - \overline{\ln \sinh 2\mathscr{C}^{(n+1)}} \\ &= \overline{|J^{(n)}|} - \overline{|J^{(n+1)}|} + \tfrac{1}{2} \overline{\ln \sinh 2 |J^{(n)}|} - \tfrac{1}{2} \overline{\ln \sinh 2 |J^{(n+1)}|}\end{aligned}$$

Since $\overline{|J^{(n)}|}$ and $\overline{\ln \sinh 2 |J^{(n)}|}$ are converging sequences in n (see proofs 1 and 8 and Section 3), the statement follows.

10. $\lim_{n\to\infty} \lambda_1^{(n)} = \lim_{n\to\infty} \overline{\ln(2 \cosh J^{(n)})}$. The expression of $\lambda_1^{(n)}$ can be written as

$$\lambda_1^{(n)} = \lim_{N\to\infty} \frac{1}{N} \overline{\ln \sum_{\{\sigma\}} \exp\left(\sum_{i=1}^{N} J_i^{(n)} \sigma_i \sigma_{i+1}\right) \cosh\left(\sum_{i=1}^{N} h_i^{(n)} \sigma_i\right)}$$

and therefore

$$\lambda_1^{(n)} \geqslant \lim_{N\to\infty} \frac{1}{N} \overline{\ln \sum_{\{\sigma\}} \exp\left(\sum_{i=1}^{N} J_i^{(n)} \sigma_i \sigma_{i+1}\right)} = \overline{\ln(2 \cosh J^{(n)})}$$

On the other hand,

$$\lambda_1^{(n)} \leqslant \lim_{N \to \infty} \frac{1}{N} \overline{\ln \sum_{\{\sigma\}} \exp\left(\sum_{i=1}^{N} J_i^{(n)} \sigma_i \sigma_{i+1} + |h_i^{(n)}| \right)} = \overline{\ln(2 \cosh J^{(n)})} + \overline{|h^{(n)}|}$$

Since $\lim_{n \to \infty} \overline{|h^{(n)}|} = 0$ (see proof 9), the statement is proved.

11. $\lim_{n \to \infty} \overline{\ln \tanh |\tilde{J}^{(n)}|} = \ln \tanh J^*$. Let us consider the difference Δ_n between $\overline{\ln \tanh |J_i^{(n)}|}$ and $\overline{\ln \tanh |\tilde{J}_i^{(n)}|}$. We recall that both couplings $J_i^{(n)}$ and $\tilde{J}_i^{(n)}$ came from the $(n-1)$th system $\{J_i^{(n-1)}, h_i^{(n-1)}\}$ via the maps, respectively, (2.6) and (4.2). By means of this Δ_n can be expressed as

$$\Delta_n = \overline{\ln \frac{a_i \cosh 2h_i^{(n-1)} + 1 + (a_i^2 \sinh^2 2h_i^{(n-1)} + b_i)^{1/2}}{a_i + \cosh 2h_i^{(n-1)} + (\sinh^2 2h_i^{(n-1)} + b_i)^{1/2}}}$$

with

$$\begin{cases} a_i \equiv e^{2J_i^{(n-1)} + 2J_{i-1}^{(n-1)}} \\ b_i \equiv 2 \cosh 2h_i^{(n-1)} e^{2J_i^{(n-1)} + 2J_{i-1}^{(n-1)}} + e^{4J_i^{(n-1)}} + e^{4J_{i-1}^{(n-1)}} \end{cases}$$

Since Δ_n is a decreasing function of b_i, one has

$$\Delta_n \leqslant \overline{\ln \frac{a_i \exp(2\,|h_i^{(n-1)}|) + 1}{a_i + \exp(2\,|h_i^{(n-1)}|)}} \leqslant 2\, \overline{|h_i^{(n-1)}|}$$

Since $\lim_{n \to \infty} \overline{|h^{(n)}|} = 0$ (proof 9), it follows that

$$\lim_{n \to \infty} \overline{\ln \tanh |\tilde{J}^{(n)}|} = \lim_{n \to \infty} \overline{\ln \tanh |J^{(n)}|} = \ln \tanh J^*$$

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